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(9) TECHNICAL REPORT (14) ARLCB-TR-77040

(6) PROBABILISTIC MODELS OF GUN-TUBE FATIGUE BASED ON A FRACTURE-MECHANICS MODEL

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(11) October 1977

(12) 52p.

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
LARGE CALIBER WEAPON SYSTEM LABORATORY
BENÉT WEAPONS LABORATORY
WATERVLIET, N. Y. 12189

AMCMS NO. 611102.H540011

DA Project No. (16) 1L161102AH54

Pron. No. 1A-7-51700-1A-M7

Approved for public release; distribution unlimited.

410 224

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ARLCB-TR-77040	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) PROBABILISTIC MODELS OF GUN-TUBE FATIGUE BASED ON A FRACTURE-MECHANICS MODEL		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) E. E. Coppola		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Benet Weapons Laboratory Watervliet Arsenal, Watervliet, N.Y. 12189 DRDAR-LCB-TL		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS AMCMS No. 611102.H540011 DA Proj. No. 1L61102AH54 PRON No. 1A-7-51700-1A-M7
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Armament Research and Development Command Large Caliber Weapon System Laboratory Dover, New Jersey 07801		12. REPORT DATE October 1977
		13. NUMBER OF PAGES 52
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Fatigue
Fracture (Mechanics)
Gun Barrels
Life Span
Mathematical Models
Monte Carlo Method
Probability
Simulation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Two probabilistic models of gun tube fatigue (those of Racicot and of Proschan and Sethuraman) have been recently developed by adding probabilistic elements to a deterministic model of fatigue failure. These probabilistic models are examined to determine if they give adequate representations when certain questionable assumptions are lifted. In addition, the deterministic model is cast into a more general probabilistic framework, and the effects of certain statistical assumptions are examined. next page

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Monte Carlo simulation studies are conducted to approximate possible distributions for gun tube fatigue lives. These generated distributions are compared to various theoretical distributions to determine their adequacy in representing fatigue data. A randomizing method of selecting distributions for material properties of the gun tube is used to give some independence from unwarranted assumptions.

Results of the simulation studies indicate that the lognormal distribution generally gives the best fit to the fatigue lives, but in most cases the lognormal distribution can be rejected by goodness-of-fit tests.

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1. INTRODUCTION

Much progress has been made in recent years in developing viable deterministic models of crack growth in gun tubes. Such models relate the amount of growth during a given number of cycles to various material properties and design parameters. Using such models one can then predict the number of cycles required for the tube to fail, provided one knows all the relevant material and design properties.

In practice, such properties vary from gun to gun and may even vary from round to round throughout the life of a single gun. Gun tube fatigue, therefore, cannot be treated as a purely deterministic process. It is necessary to allow a place for random elements which must be analyzed statistically rather than deterministically. The Army has recognized this fact by stating gun tube fatigue requirements in statistical terms.

Racicot[1] and Proschan and Sethuraman[2] have devised probabilistic models of gun tube fatigue by adding stochastic elements into an existing deterministic model of fatigue. It is the aim of this paper to examine this probabilistic approach in somewhat more detail and to examine somewhat more closely the effects of various assumptions and

[1]Racicot, R. L., "A Probabilistic Model of Gun Tube Fatigue," Watervliet Arsenal Technical Report No. ARLCB-TR-77029 Watervliet, NY May 1977. All further references to Racicot are to this report.

[2]Proschan, F., and Sethuraman, J., "A Probabilistic Model for Initial Crack Size and Fatigue Life of Gun Barrels," Florida State University Technical Report No. M394, The Florida State University, Tallahassee, Florida, December 1976.

methods of these authors on their conclusions. Racicot has already stated the benefits to be derived from a closer study of the probabilistic aspect of gun fatigue; we shall not repeat them here.

2. A FRACTURE MECHANICS MODEL OF CRACK GROWTH CONSIDERED STOCHASTICALLY

The deterministic model mentioned above was developed by Throop and others[3] at Watervliet Arsenal based on the Paris expression for rate of crack growth and experimental results. The rate of crack growth is approximated by:

$$\frac{db}{dN} = \frac{(\Delta K)^m}{M} \tag{1}$$

where: b = crack depth

ΔK = range of variation of stress intensity factor K over one cycle

N = number of cycles

m = a parameter dependent on material properties and stress intensity. Usually m is between 2 and 4.

M = a parameter dependent on material properties.

In this model, ΔK and M are given by:

$$\Delta K = \alpha S\sqrt{\pi b} \tag{2}$$

$$M = EK_{Ic}\sigma_y/C \tag{3}$$

where:

S = maximum hoop stress at the bore

α = a parameter depending on crack shape and the residual stresses introduced by the autofrettage process

[3] Throop, J. F., and Miller, G. A., "Optimum Fatigue Crack Resistance," ASTM Special Technical Publication 467, Philadelphia, PA, 1970, pp. 154-168.

E = Young's modulus

K_{Ic} = fracture toughness for a crack in a tangential stress field

σ_y = yield strength

C = a parameter varying with m to maintain dimensional homogeneity and possibly depending on material properties.

Combining (1), (2) and (3) gives:

$$\frac{db}{dN} = Gb^{k+1} \tag{4}$$

where: $k = \frac{m}{2} - 1$

$$G = \frac{C(\alpha S\sqrt{\pi})^m}{E\sigma_y K_{Ic}} \tag{5}$$

Racicot assumes that G and k are essentially constant during tube life, so that he can solve the differential equation (4) by simple integration to give*:

$$N = \frac{1}{Gk}(b_o^{-k} - b^{-k}) \tag{6}$$

where b_o is an initial crack depth, assumed to be present after a few rounds of firing.

A different line of reasoning, not assuming the constancy of G, can also be used. (This line of reasoning is used by Cramer[4] in a similar problem.) Assume that k is essentially constant over tube life. Equation (4) may be rewritten:

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*Racicot actually used $N-N_i$ where we have used N. N_i is the number of cycles required to crack initiation. In practice N_i is small enough to be ignored.

[4]Cramer, H., Mathematical Methods of Statistics, Princeton University Press, Princeton, NJ, 1946, p. 219.

$$b_j - b_{j-1} = G_j b_{j-1}^{k+1} \tag{7}$$

where b_j = the crack length after the jth cycle.

G_j = the value of G immediately preceding the jth cycle.

From equation (7), we have:

$$G_j = \frac{b_j - b_{j-1}}{b_{j-1}^{k+1}} \tag{8}$$

Summing over j gives:

$$\sum_{j=1}^{N} G_j = \sum_{j=1}^{N} \frac{b_j - b_{j-1}}{b_{j-1}^{k+1}} \tag{9}$$

If $b_{j+1}-b_j$ tends to be relatively small, then the sum on the right-hand side of (9) can be approximated by:

$$\int_{b_o}^{b_N} \frac{db}{b^{k+1}} = \frac{1}{k} (b_o^{-k} - b_N^{-k})$$

And so at least approximately:

$$\sum_{j=1}^{N} G_j = \frac{1}{k} (b_o^{-k} - b_N^{-k}) \tag{10}$$

Assuming that G is constant over tube life means that $G_j = G$ for all j and hence:

$$\sum_{j=1}^{N} G_j = NG \tag{11}$$

Combining (10) and (11) then gives equation (6) for constant G.

From a probabilistic viewpoint, equation (10) is very nice. First, it shows that crack growth depends probabilistically on four things:

a. The distribution of the initial crack depth b_o.

b. The distribution of the rate exponent k.

c. The distribution of G_1, which is the value of G immediately following the forming of the initial crack and preceding the first cycle.

d. The manner in which G probabilistically varies from cycle to cycle.

Second, equation (10) involves a sum of random variables which suggests that if the G_j's are sufficiently well-behaved, some classicial results such as the central limit theorems may be exploited.

Also note that the argument used to derive (10) does not depend in any way on the exact material properties or design variables that go into G, nor does it require any independence (mathematical, physical or statistical) of G and b.

The authors cited above all assume that G does not vary from cycle to cycle. (Racicot did allow S to vary for some of his calculations.) We shall consider the effect of this assumption next.

3. EFFECTS OF CYCLE-TO-CYCLE VARIATIONS IN THE MODEL PARAMETERS

There is no reason to suppose that the G_j's are actually constant from cycle to cycle. However, if the probability distribution of $\sum_{j=1}^{N} G_j$ is close to that of NG_1, in some suitable sense, then NG_1 can effectively replace $\sum_{j=1}^{N} G_j$ in equation (10).

Since fatigue failures typically occur only after thousands of cycles, it is quite legitimate to assume that in equation (10), N is large. This allows us to make good use of the central limit theorem and the Laws of Large Numbers.

We will examine the probabilistic behavior of b_N for large N under some assumption as to how G_j varies from cycle to cycle. For some of these assumptions, we will show that the asymptotic behavior (that is, the behavior as N approaches infinity) is not much different from the behavior when the G_j's are assumed not to vary from cycle to cycle. In other cases, the asymptotic behavior can be quite different.

The various sets of assumptions listed below are not meant to be exhaustive. They were selected simply because they are the sort of assumptions that spring most readily to a statistician's mind. They are also the sort of assumptions that are used, often without conscious realization, by non-statisticians who employ "cook-book" formulas from the standard texts. In the discussion below, we will employ "error terms" denoted by $\varepsilon_1, \varepsilon_2 \ldots$ We assume that these are independent and identically distributed with common mean μ_ε and common standard deviation σ_ε, where $0 < \sigma_\varepsilon < \infty$. We also let μ_G and σ_G be the mean and standard deviation, respectively of G_1. Let $S_N = G_1 + - + G_N$. Then, the problem is to determine the asymptotic behavior of $S_N - NG_1$.

Case I: $G_j = \varepsilon_j$

This assumption implies that $G_1, G_2 \ldots$ are statistically independent. However, since $G_1, G_2 \ldots$ are functions of material and design parameters, which should not change that radically from cycle to cycle, the independence of the G's would seem unlikely. We shall not consider this case further.

Case II: $G_j = G_1 + \varepsilon_{j-1}$ for $j > 1$

This is a simple linear regression model. In this case, $S_N = NG_1 + \varepsilon_1 + \ldots + \varepsilon_{N-1}$. The Laws of Large Numbers[4] then imply that $(S_N - NG_1)/(N-1)$ approaches μ_ε in probability. In particular if $\mu_\varepsilon = 0$, then distribution of S_N may be approximated by that of NG_1.

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[4]Cramer, H., Mathematical Methods of Statistics, Princeton University Press, Princeton, NJ, 1946, p. 497.

How good the approximation will be depends on σ_ε. In fact it follows from the Central Limit Theorem that for large N,

$$S_N = NG_1 + (N-1)\mu_\varepsilon + (N-1)^{1/2}\sigma_\varepsilon U_N$$

where U_N has approximately a standard normal distribution. When $\mu_\varepsilon = 0$ and σ_ε is sufficiently small, the deviation $S_N - NG_1$ will tend to be close to zero for large, but not too large N.

Case III: $G_j = \varepsilon_{j-1}G_1$ for $m > 1$.

In this case, $S_N - NG_1 = (\varepsilon_1 + \ldots + \varepsilon_{N-1} - N)G_1$. The Laws of Large Numbers imply that $(S_N - NG)/(N-1)$ approaches $(\mu_\varepsilon - 1)G_1$ in probability. If $\mu_\varepsilon = 1$, this would suggest that again the distribution of S_N may be approximated by that of NG_1. From the Central Limit Theorem,

$$S_N = NG_1 + (N-1)(\mu_\varepsilon - 1)G_1 + (N-1)^{1/2}\sigma_\varepsilon G_1 U_N$$

where U_N has approximately a standard normal distribution for large N. Again, the goodness of the approximation depends on σ_ε.

Case IV: $G_j = G_{j-1} + \varepsilon_{j-1}$ for $j > 1$.

This looks somewhat like Case II. In fact, it is quite different. In this case:

$$G_j = G_1 + \varepsilon_1 + \ldots + \varepsilon_{j-1}$$

Then

$$S_N = NG_1 + \sum_{j=1}^{N-1} (N-j)\varepsilon_j$$

It can be shown (see Appendix) that as $N \to \infty$, the distribution of:

$$\frac{S_N - NG_1 - N(N-1)\mu_\varepsilon/2}{[N(N-1)(2N-1)\sigma_\varepsilon^2/6]^{1/2}}$$

is asymptotically standard normal. One could again say that S_N is approximated by NG_1 when $\mu_\varepsilon = 0$, but the approximation may not be good. In cases II and III, σ_ε was multiplied by a factor $(N-1)^{1/2}$. Here , however, σ_ε is multiplied by a factor approximately $(N-1)^{3/2}$, which is larger and will require much smaller σ_ε for the approximation to be plausible.

There is, however, another feature of this model that is rather intriguing. Namely, it can be shown that (Appendix):

$$\frac{S_N - N\mu_G - N(N-1)\mu_\varepsilon/2}{[N^2\sigma_G{}^2 + N(N-1)(2N-1)\sigma_\varepsilon{}^2/6]^{1/2}}$$

has an asymptotic standard normal distribution. Note that G_1 appears nowhere in this expression, only the first two moments of G_1 appear in the expression and this result holds whatever the distribution of G_1, provided σ_G is finite.# Such a result is very nice from a statistical viewpoint. However, we shall not pursue its implications in this report.

4. THE ROBUSTNESS OF RACICOT'S MODEL

As previously mentioned, Racicot assumed that the G_j's remain constant throughout the life of the tube. However, he assumed that the material and design parameters entering into G varied from tube to tube. These assumptions will be adopted here. Equation (6) then

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#In fact similar results not involving G_1 or its moments in any way can also be derived. See Appendix.

holds for any particular tube, but G, k and b_o will vary from tube to tube. Equation (5) expresses G as a function of various material parameters. Each of these parameters is assumed to vary.

Fatigue failure occurs when the crack either penetrates through the tube or reaches a critical depth where unstable growth occurs. The critical depth b_c is given by:

$$b_c = \frac{A}{\pi} \left(\frac{K_{Ic}}{\alpha S}\right)^2 \tag{12}$$

where A is an empirical constant that accounts for differences in crack shape and in the specimens used to determine K_{Ic}.

Fatigue failure then occurs on the earliest cycle N such that $b_N \geq \min(B, b_c)$ where B is the wall thickness of the tube. Since N is usually large, we can ignore the fact that N must be an integer. This gives

$$N = \frac{1}{kG} (b_o^{-k} - b^{-k}) \tag{13}$$

where $b = \min(b_c, B)$.

If the probability distributions of b_o and of each of the model parameters were known, then it would be relatively simple to approximate the distribution of N through Monte Carlo simulation. However, none of these distributions are known. Racicot therefore assumed that the distributions of each of the relevant random variables followed some distribution of known mathematical form and then estimated the parameters of the distributions from test data gathered by Davidson and others. He then generated the distribution of N by Monte Carlo simulation using equation (13).

Certain methods used by Racicot are, however, open to criticism and could possibly weaken his conclusions.

First, Racicot assumed that each of the model parameters had the same type of distribution. Specifically, he considered 3 cases: all of the parameters had normal distributions; all had Weibull distributions; and all had lognormal distributions. It may in fact be the case that all the model parameters have the same type of distributions, at least approximately. But there is at this time no real reason to believe so. Even if they did have the same type of distribution, there is no reason to assume that this type would be either normal, lognormal or Weibull.

Second, Racicot compared his Monte-Carlo distribution with various possible candidate distributions by using the Kolmogorov-Smirnov (KS) statistic[5] with the parameters of the candidate distribution estimated by the moment-matching method[4]. However, for the lognormal and Weibull distributions, the moment-matching method is not the best method to use. Further, the KS statistic is known to be inferior in some ways to other goodness-of-fit statistics[6]. How would Racicot's conclusions fare under better methods of estimation and other goodness-of-fit tests?

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[4] Cramer, H., Mathematical Methods of Statistics, Princeton University Press, Princeton, NJ, 1946, p. 497.

[5] Lindgren, B. W., Statistical Theory, The Macmillan Company, New York, NY, 1970, p. 329.

[6] Stephens, M. A., "EDF Statistics for Goodness of Fit and Some Comparisons," Journal of the American Statistical Association, 69, (September 1974), pp. 730-737.

Third, other candidate distributions could be considered.

Fourth, Racicot assumed that most of the model parameters were statistically independent. Again, there is no real reason to assume that this is so.

The next sections of this report will describe the efforts of the author to fill in these gaps.

5. INDEPENDENT MODEL PARAMETERS WITH RANDOMLY SELECTED DISTRIBUTIONS

Racicot used data from nine 105mm M137A1 tubes tested by Throop and others as a base for estimating the distributions of the model parameters. The present author will continue to use these data. α, K_{Ic}, E, b_o and m are assumed to be random variables with means and standard deviations as given in Table 1.

TABLE 1. MEANS AND STANDARD DEVIATIONS OF MODEL PARAMETERS FOR 105MM M137A1 TUBES

Parameter	Mean	Standard Deviation
α	0.877	0.050
K_{Ic} (KSI$\sqrt{in}$)	103	11.5
E (KSI)	30,000	300
b_o (in)	0.02	0.001
m	3.5	0.1

σ_y was given by the empirical relationship: $\sigma_y = 334 - 1.39\ K_{Ic}$. The other parameters (S,A,B,C) were assumed constant. α, K_{Ic}, E, b_o and m were assumed to be statistically independent.

The distribution of N, the fatigue failure time, was then approximated by computer simulation. For each run of the simulation program through the computer, the program itself chose, at random, a type of distribution of each of the five random model parameters. The choice of a distribution type for each of the model parameter was independent of the choice of type for any of the others. The rationale behind this method is: The author wanted to check that Racicot's assumption of the same distribution type for each of the five parameters did not somehow bias his results. However, it is clearly impossible to try all possible different combinations of types. By instructing the computer to select distribution types at random, the author hoped to obtain a representative sampling of possible combinations. The possible distribution types available for selection were: normal, lognormal, uniform, extreme value, Weibull and 2-parameter exponential.

Once the distribution type was selected, the actual parameters of the distribution were estimated by the method of moments from the means and standard deviations given above.

Ten thousand replicates of N were then generated for each run by generating ten thousand sets of replicates of α, K_{Ic}, E, b_o and m and calculating a corresponding N for each set of replicates using equation (13). Because of computer space limitations, it was

necessary to round each N to the nearest 10 cycles; however, this should have no significant effect.

The distribution of N generated by this method was then compared to each of nine candidate parametric distribution families. For uniformity, we will characterize each of the statistical parameters as either a location parameter denoted by η, a shape parameter denoted by β or a scale parameter denoted by Θ. Also, $\Phi(x)$ represents the standard normal cumulative distribution function, and $F(x)$ represents the cumulative distribution function (cdf). The nine families are:

Normal

$$F(x) = \Phi(\frac{x-\eta}{\Theta}) \qquad \text{all } x$$

Lognormal

$$F(x) = \Phi[\frac{1}{\beta} \ln(\frac{x}{\Theta})] \qquad x > 0$$

Extreme value

$$F(x) = 1-\exp[-e^{(x-\eta)/\Theta}] \qquad \text{all } x$$

Weibull

$$F(x) = 1-\exp[-(x/\Theta)^{\beta}] \qquad x > 0$$

1-parameter Exponential

$$F(x) = 1-\exp(-x/\Theta) \qquad x > 0$$

2-parameter Exponential

$$F(x) = 1-\exp(\frac{\eta-x}{\Theta}) \qquad x > \eta$$

Double-Exponential (Fisher-Tippett Type I)

$$F(x) = \exp[-e^{(\eta-x)/\Theta}] \qquad \text{all } x$$

Inverse-Weibull

$$F(x) = \exp[-(\Theta/x)^{\beta}] \qquad x > 0$$

Birnbaum-Saunders

$$F(x) = \Phi[\frac{1}{\beta}(\frac{x}{\Theta})^{1/2} - \frac{1}{\beta}(\frac{x}{\Theta})^{-1/2}] \qquad x > 0$$

For all of these, $F(x) = 0$, except where otherwise noted.

The 3-parameter versions of the Weibull, lognormal and inverse-Weibull will not be considered here. These distributions are very difficult to deal with in practice with the limited data found in fatigue-life problems. The whole point of this exercise is to give the applied statistician some method he can use in real-life problems. At the present time, the 3-parameter versions simply do not provide such a method.

It is also generally true that the more parameters one adds to a distribution, the better one can fit arbitrary sets of data to it, even if the distribution has little justification on empirical grounds. Multi-parameter Weibull distributions have given good fits to some fatigue data in the past probably because they had enough parameters to give a wide variety of shapes[7]. By limiting ourselves to two-parameter distributions, we will hopefully arrive at something that has some relation to underlying physical process and avoid labeling a distribution as good fitting only because it has enough parameters to smooth out the rough spots.

For each of the families, approximation of the statistical parameters (η, Θ, and β) used the moment-matching method for the normal, extreme-value, 1-parameter exponential, double-exponential and Birnbaum-Saunders. Moment-matching on the logarithms of the

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[7] Little, R. E., and Jebe, E. H., The Statistical Design of Fatigue Experiments, New York, John Wiley & Sons, 1975, p. 221.

fatigue lives was used for estimation for the lognormal, Weibull and inverse-Weibull. The 2-parameter exponential used minimum-variance unbiased estimators.

When estimators of the statistical parameters are substituted for the actual parameters in the appropriate expression for $F(x)$, one obtains an estimate $\hat{F}(x)$ of $F(x)$. The function $\hat{F}(x)$ can then be compared to the Monte Carlo distribution of N to test goodness of fit. We have used three different goodness-of-fit statistics. They are as follows:

Kolmogorov-Smirnov (KS)

$$D = \max_{1 \leq i \leq M} \left[\frac{i}{M} - Z_i, Z_i - \frac{i-1}{M} \right]$$

Cramer-von-Mises (CVM)

$$W^2 = \frac{1}{12M^2} + \frac{1}{M} \sum_{i=1}^{M} (Z_i - \frac{2i-1}{2M})^2$$

Anderson-Darling (AD)

$$A^2 = -M - \frac{1}{M} \sum_{i=1}^{M} (2i-1) \ln(\frac{Z_i}{1-Z_{M+1-i}})$$

where $M = 10,000$

N_i = the ith smallest fatigue life from the Monte Carlo distribution ($1 \leq i \leq 10,000$)

$Z_i = \hat{F}(N_i)$

For all three of these goodness-of-fit statistics, the smaller the value, the better the fit.

This whole procedure was repeated seven times, giving seven different possible distributions for the fatigue lives. Details are given here for three of them, which are representative of the type of

results obtained. Table 2 shows the distribution types selected for each of the five varying model parameters. Tables 3, 4 and 5 show the results of the goodness-of-fit tests.

For run #1, lognormal gives the best fit, followed very closely by Birnbaum-Saunders. These results are reversed for run #2. However, it should be noted that the cdf's of the lognormal and the Birnbaum-Saunders are so close as to be practically indistinguishable. However, the fit is not of the best. In fact, all three goodness-of-fit statistics for the lognormal are significantly too large at a significance level of 15% or smaller [6].

For run #3, the inverse-Weibull gives an excellent fit. Unfortunately, tables for testing the goodness-of-fit statistics in this case are not available.

Figures 1, 2 and 3 show histograms of the Monte Carlo fatigue life simulations. Figures 4, 5 and 6 show the cdf of the Monte Carlo distribution along with the lognormal, Weibull, inverse-Weibull and 2-parameter exponential distributions used to approximate it.

6. CORRELATED MODEL PARAMETERS

Up to now, the five model parameters α, K_{Ic}, E, b_o and m have been considered statistically independent. There is no real reason for believing that they actually are.

[6] Stephens, M. A., "EDF Statistics for Goodness of Fit and Some Comparisons," Journal of the American Statistical Association, 69, (September 1974), pp. 730-737. (Stephen's tables for Case 2 normal apply unchanged to the lognormal distribution with the type of estimators we have been using).

TABLE 2. DISTRIBUTION TYPES OF THE MODEL PARAMETERS

	Run #1	Run #2	Run #3
α	uniform	uniform	extreme-value
K_{Ic}	normal	normal	uniform
E	uniform	lognormal	lognormal
b_o	uniform	uniform	extreme-value
m	extreme-value	normal	lognormal

TABLE 3. GOODNESS-OF-FIT RESULTS

Run #1

Distribution Type	Parameters Location	Shape	Scale	Goodness-of-Fit Statistics KS	CVM($x10^4$)	AD
Normal	11737	-	2570	0.072	16.6	10.6
Lognormal	-	4.594	11463	0.051	9.5	6.2
Extreme-Value	12894	-	2004	0.13	61.9	48.9
Weibull	-	5.893	12642	0.10	38.7	29.5
Exponential 1-parameter	-	-	11738	0.47	594.7	282.5
Exponential 2-parameter	5630	-	6108	0.30	287.3	149.9
Double-Exponential	10581	-	2004	0.063	16.5	10.8
Inverse-Weibull	-	5.893	10393	0.087	28.0	20.2
Birnbaum-Saunders	-	0.2178	11466	0.052	9.6	6.2

TABLE 4. GOODNESS-OF-FIT RESULTS

Run #2

Distribution Type	Parameters Location	Shape	Scale	Goodness-of-Fit Statistics KS	CVM($\times 10^4$)	AD
Normal	11689	-	2825	0.062	12.5	7.9
Lognormal	-	4.182	11359	0.035	3.8	2.4
Extreme-Value	12960	-	2202	0.129	58.4	$>10^9$
Weibull	-	5.363	12650	0.090	29.9	23.9
Exponential 1-parameter	-	-	11689	0.445	562.6	268.5
Exponential 2-parameter	5310	-	6379	0.281	268.8	140.2
Double-Exponential	10418	-	2202	0.044	8.1	5.22
Inverse-Weibull	-	5.363	10200	0.080	21.9	16.69
Birnbaum-Saunders	-	0.2401	11361	0.035	3.7	2.3

TABLE 5. GOODNESS-OF-FIT RESULTS

Run #3

Distribution Type	Parameters Location	Shape	Scale	Goodness-of-Fit Statistics KS	CVM($x10^4$)	AD
Normal	12627	-	5834	0.153	99.1	$>10^9$
Lognormal	-	2.854	11768	0.068	17.0	10.5
Extreme-Value	15253	-	4549	0.220	179.1	$>10^9$
Weibull	-	3.660	8324	0.137	68.9	$>10^9$
Double Exponential	10001	-	4549	0.120	44.6	30.1
Exponential 1-parameter	-	-	12627	0.399	425.5	206.1
Exponential 2-parameter	4950	-	7677	0.229	156.4	85.9
Inverse-Weibull	-	3.660	10051	0.011	0.2	0.2
Birnbaum-Saunders	-	0.4545	11445	0.117	40.9	28.2

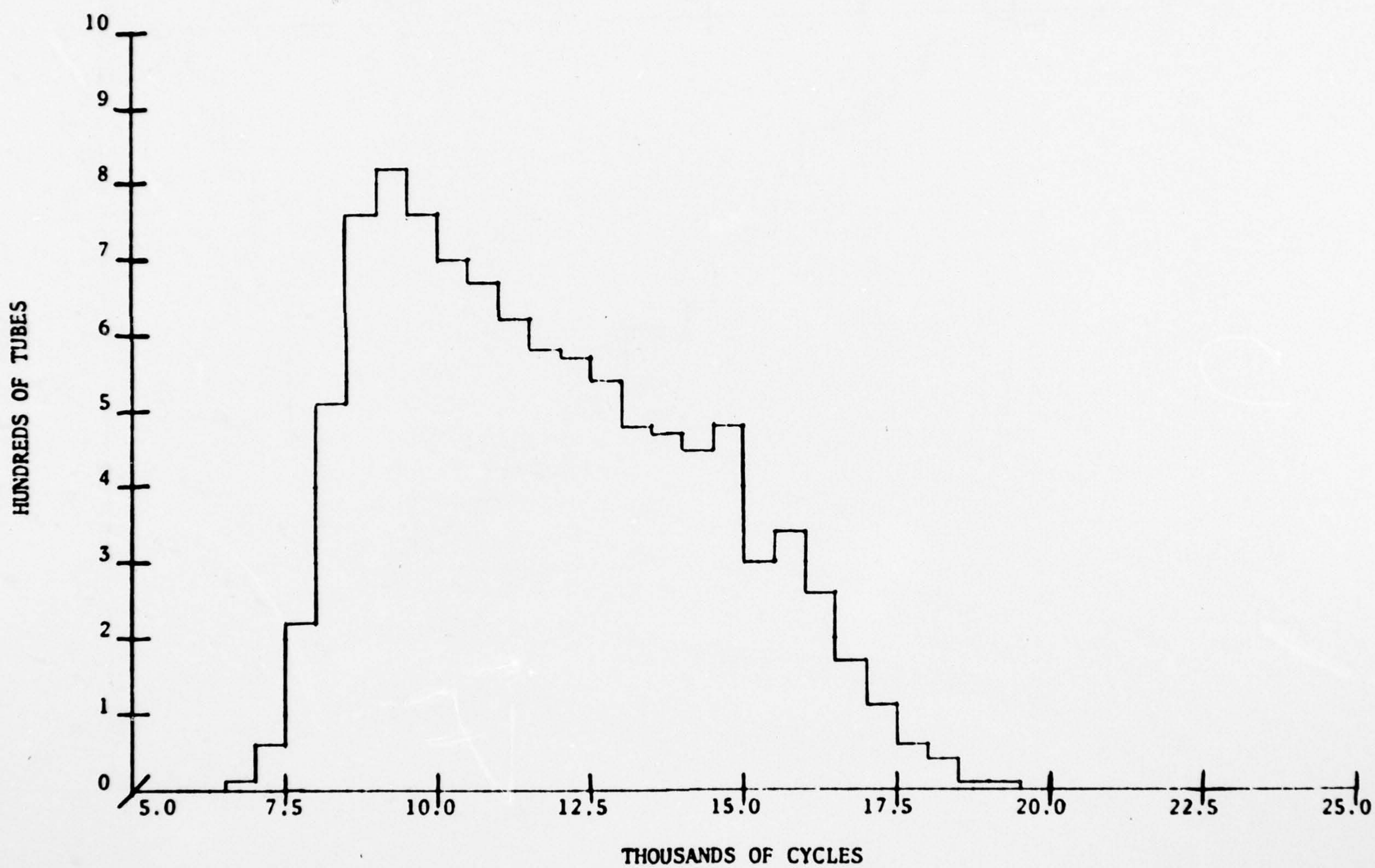

Figure 1. Frequency histogram - Run #1.

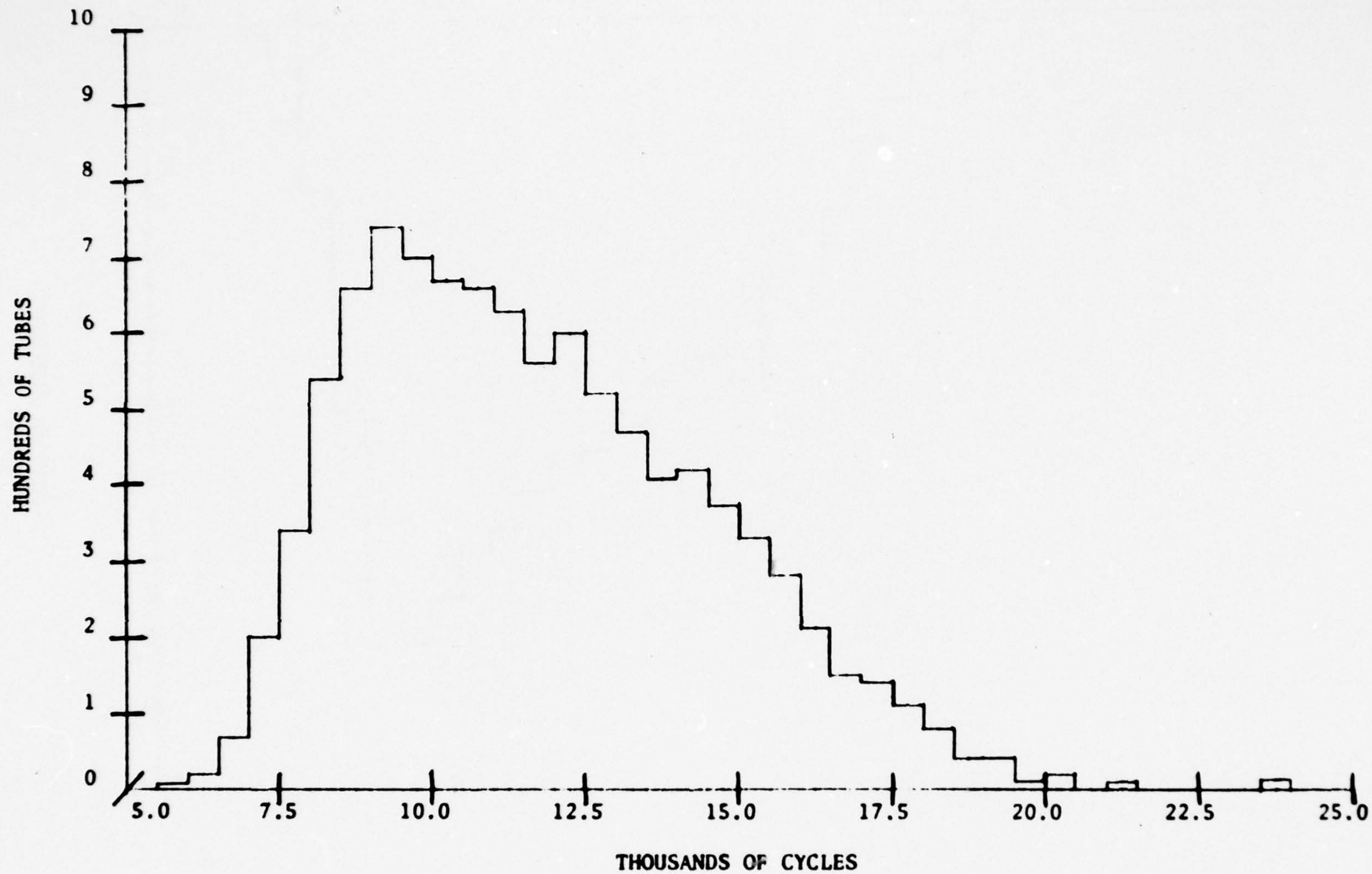

Figure 2. Frequency histogram - Run #2.

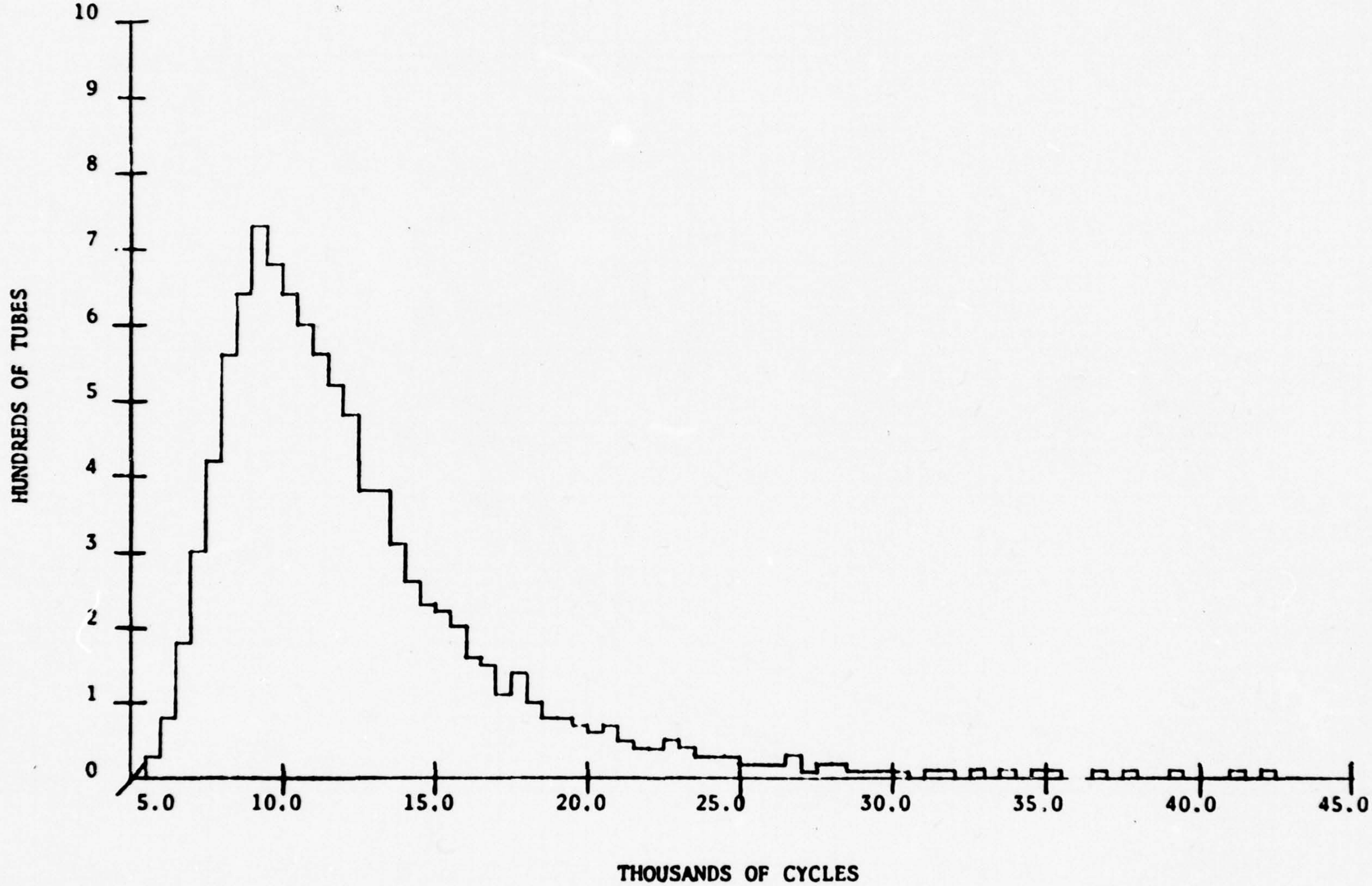

Figure 3. Frequency histogram - Run #3.

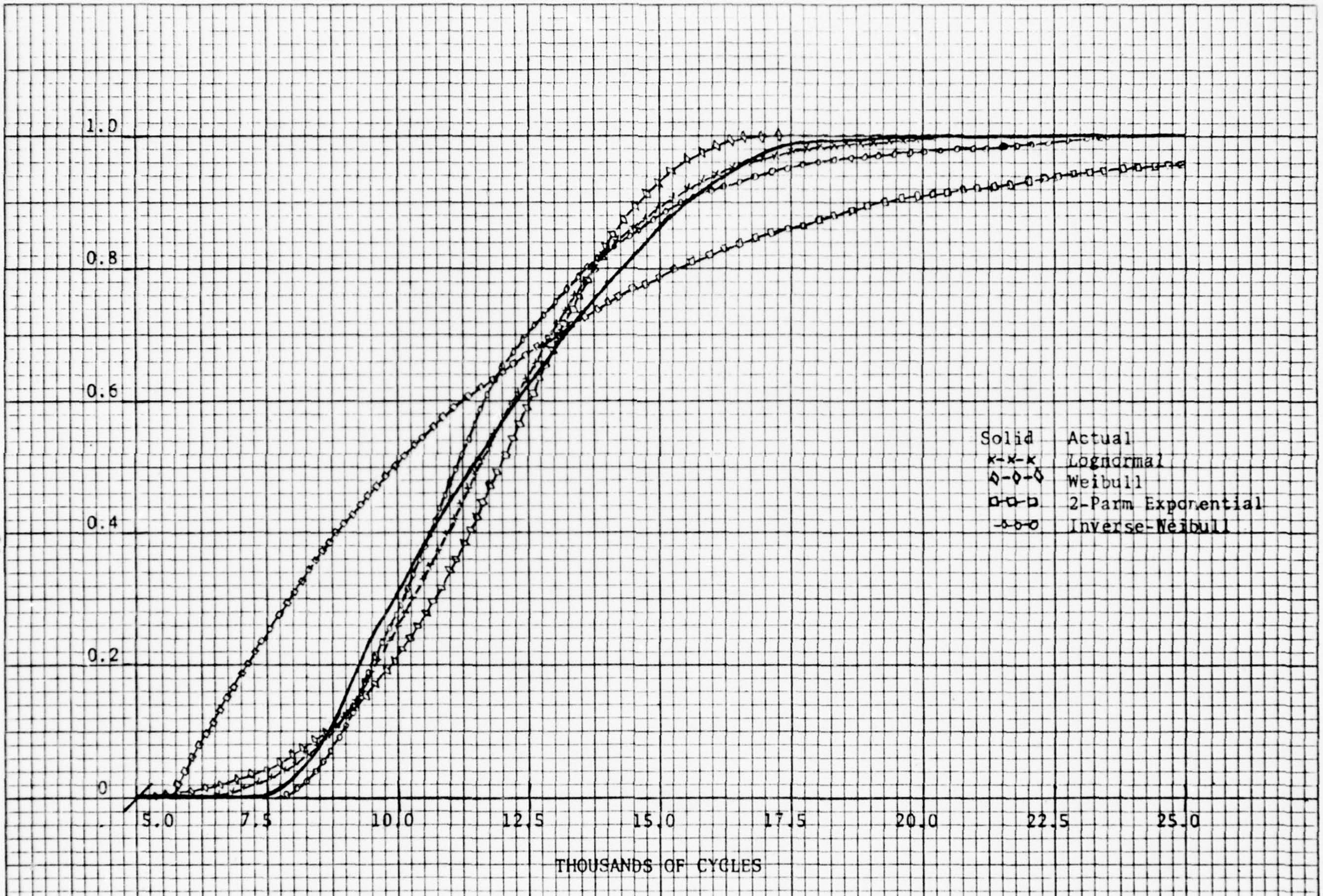

Figure 4. Cumulative distribution - Run #1.

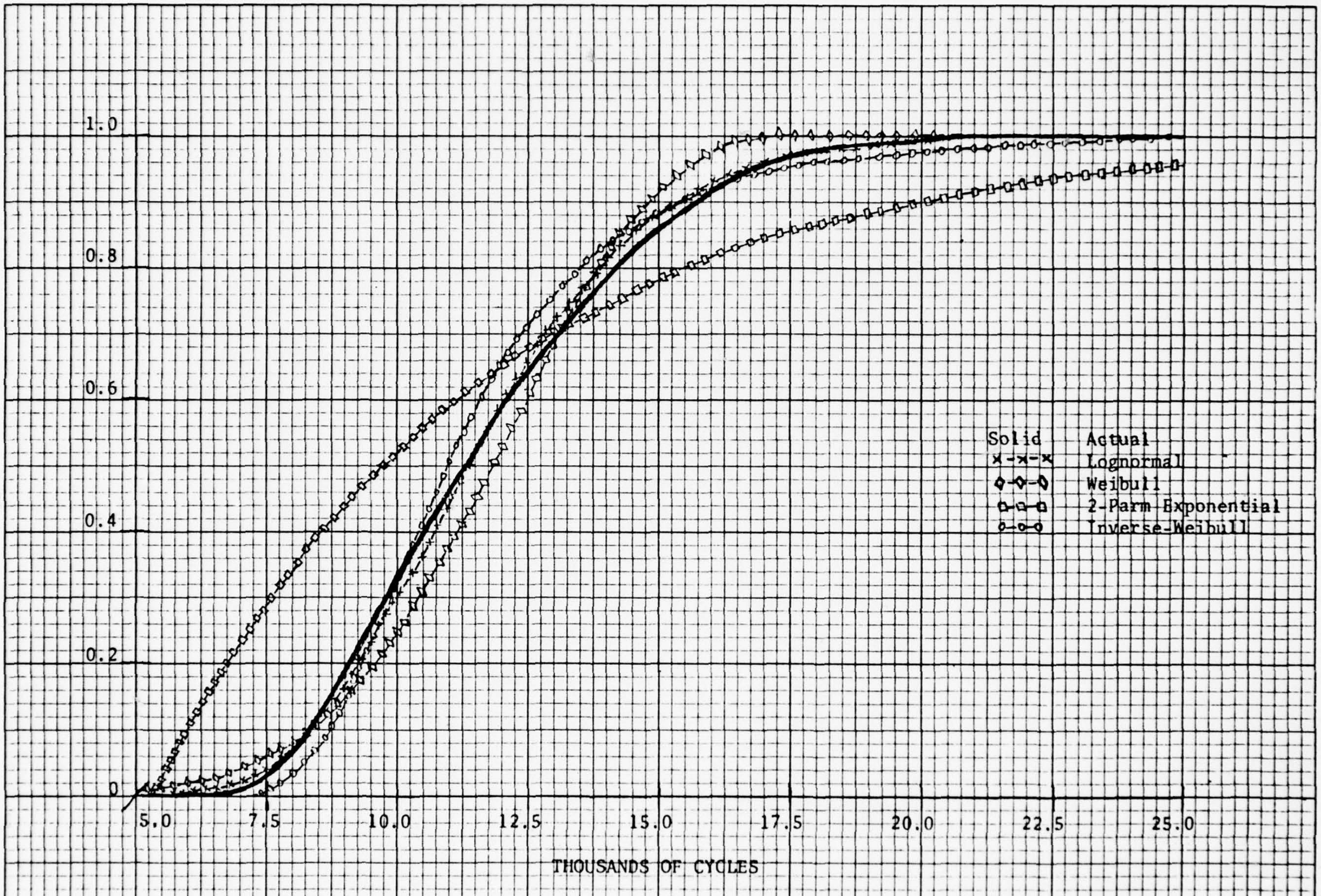

Figure 5. Cumulative distribution - Run #2.

Figure 6. Cumulative distribution - Run #3.

Two more runs were performed as described in the last section. However, in these two runs, all of the five model parameters were taken to be normal. On each run, the program selected at random a correlation coefficient for each pair of the five model parameters.

The results for these two runs (Labeled NC1 and NC2) are shown in Tables 6 through 8 and in Figures 7 through 10. For NC1, the lognormal gives an excellent fit. None of the three goodness-of-fit statistics are significant at any level below 15%. For NC2, the double exponential gives the best fit, followed along by the lognormal. The lognormal does not give too bad a fit. The KS statistic is significant at 15% and below, but neither the CVM nor the AD statistics are significant.

7. THE EFFECTS OF THE MODEL PARAMETER α

Racicot concluded that the model parameter α contributed more than any of the other parameters to the variability in fatigue life. A stronger statement can be made. α is not only the major source of variability in fatigue life, but it also is the major determinant of the very shape of the fatigue life distribution. It would be an exaggeration to say that E, K_{Ic}, b_o and m are all second-order effects compared to α; however, it would not be a gross exaggeration.

Figures 11, 12 and 13 show the cdf of fatigue life obtained when only α is assumed to vary from tube to tube. (The other model parameters are kept constant at their mean values.) The dotted lines show the cdf's obtained when all five of the model parameters α, K_{Ic}, E, b_o and m are assumed to vary from tube to tube. Figure 11 shows

TABLE 6. CORRELATION COEFFICIENTS FOR THE MODEL PARAMETERS

Model Parameters	NC1	NC2
α, K_{Ic}	0.719	-0.406
α, E	-0.498	0.639
α, b_o	0.111	0.408
α, m	0.097	-0.610
K_{Ic}, E	-0.937	0.391
K_{Ic}, b_o	0.175	0.437
K_{Ic}, m	0.339	-0.415
E, b_o	-0.357	0.609
E, m	-0.372	-0.999
b_o, m	0.462	-0.584

TABLE 7. GOODNESS-OF-FIT RESULTS

Run NC1

Distribution	Parameters			Goodness-of-Fit Statistics		
Type	Location	Shape	Scale	KS	CVM(x10^4)	AD
Normal	11620	-	2262	0.043	6.9	4.5
Lognormal	-	5.224	11408	0.007	0.08	0.08
Extreme-Value	12638	-	1763	0.114	47.5	$>10^9$
Weibull	-	6.700	12435	0.075	21.0	$>10^9$
Exponential 1-parameter	-	-	11620	0.469	649.0	305.2
Exponential 2-parameter	5169	-	6451	0.349	406.0	200.5
Double Exponential	10602	-	1763	0.033	3.52	2.7
Inverse-Weibull	-	6.700	10466	0.071	18.1	14.9
Birnbaum-Saunders	-	0.1938	11406	0.009	0.2	0.2

TABLE 8. GOODNESS-OF-FIT RESULTS

Run NC2

Distribution Type	Parameters Location	Shape	Scale	Goodness-of-Fit Statistics KS	CVM($x10^4$)	AD
Normal	11879	-	3210	0.068	15.9	$>10^9$
Lognormal	-	3.867	11482	0.016	0.5	0.3
Extreme-Value	13323	-	2502	0.138	66.9	$>10^9$
Weibull	-	4.960	12899	0.084	25.6	$>10^9$
Exponential 1-parameter	-	-	11879	0.426	542.0	258.8
Exponential 2-parameter	4389	-	7489	0.310	322.0	162.4
Double-Exponential	10434	-	2502	0.010	0.3	0.2
Inverse-Weibull	-	4.960	10220	0.061	14.7	12.7
Birnbaum-Saunders	-	0.2681	11466	0.018	0.8	0.7

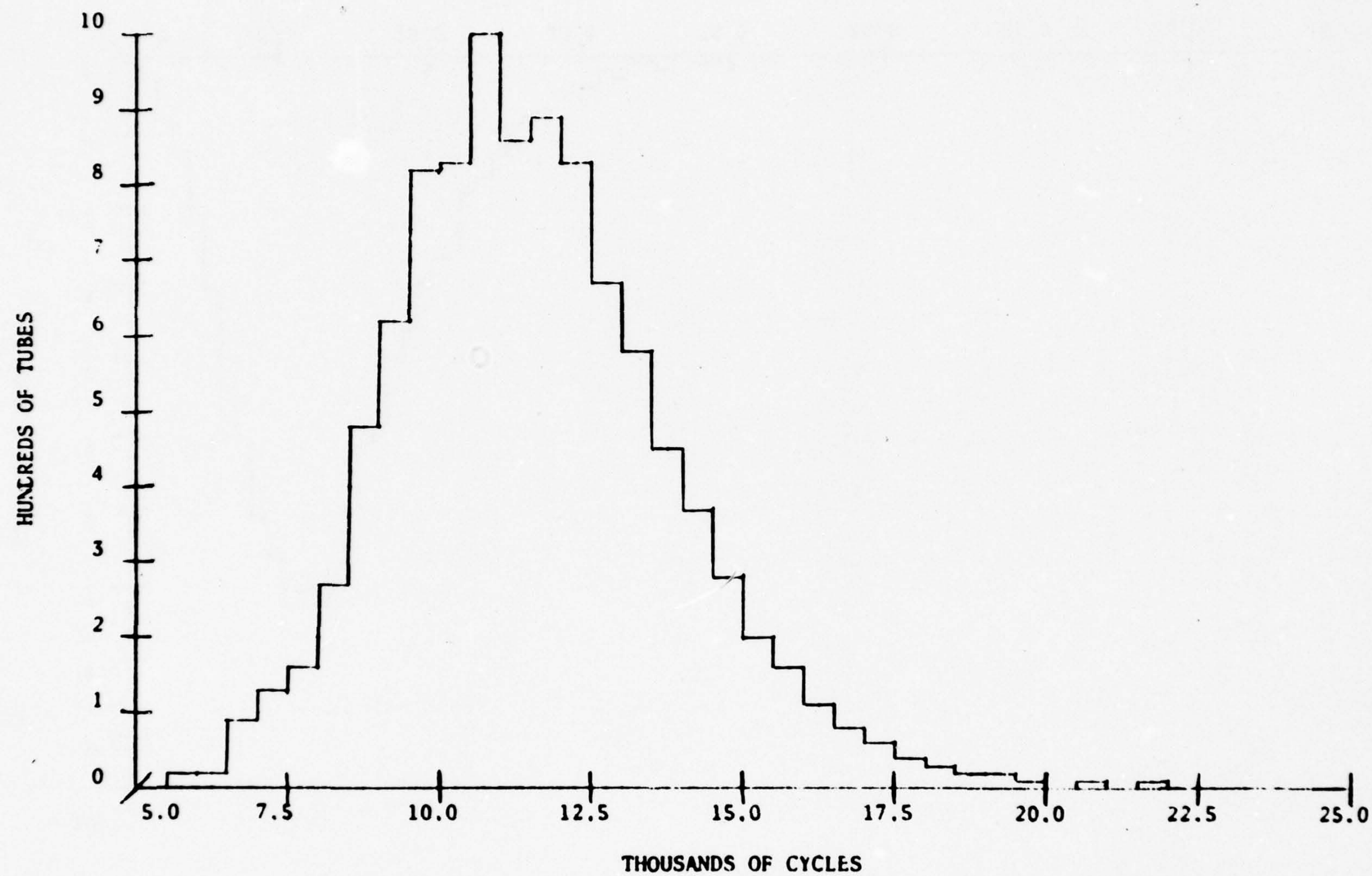

Figure 7. Frequency histogram - Run NC1.

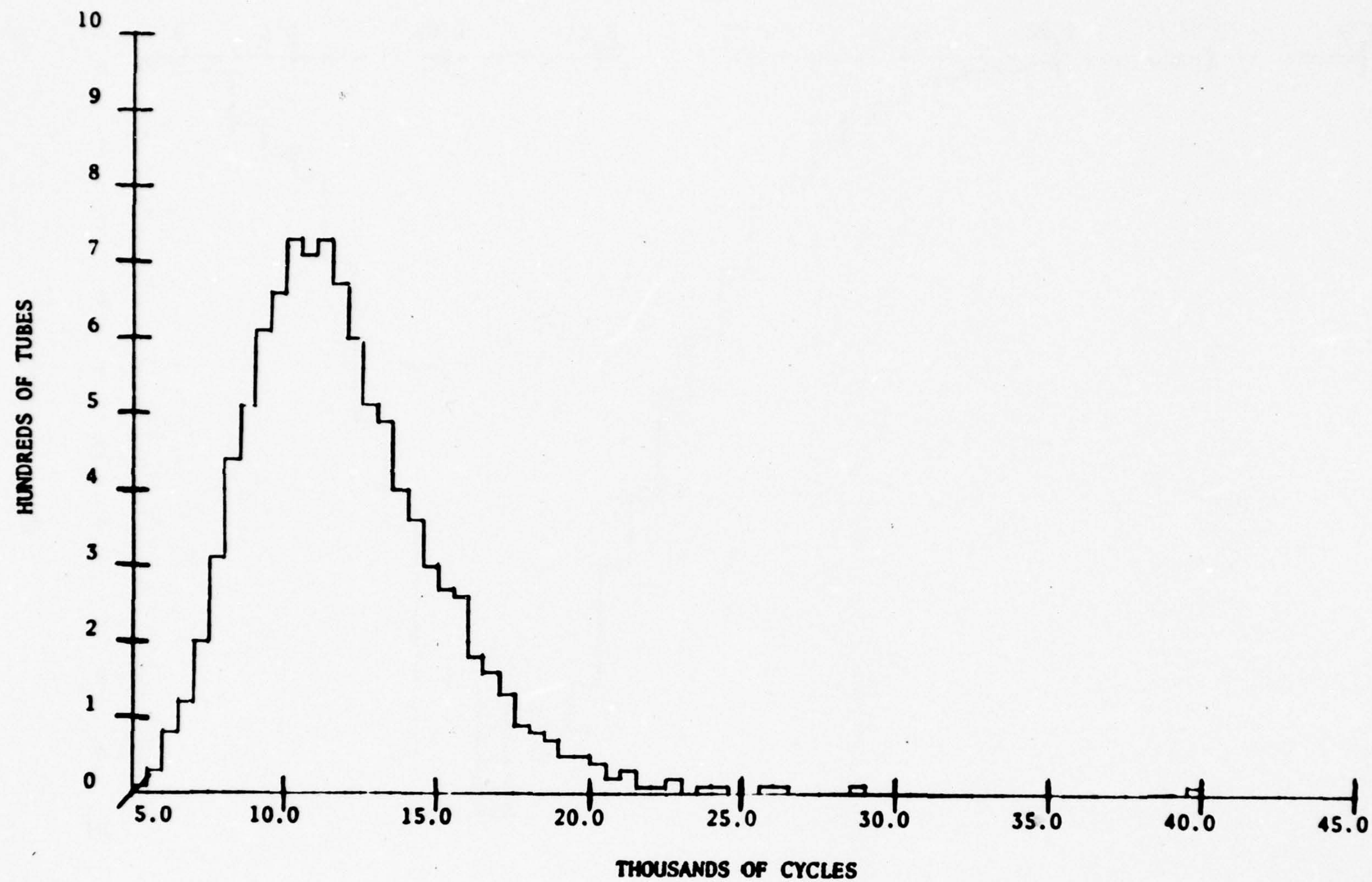

Figure 8. Frequency histogram - Run NC2.

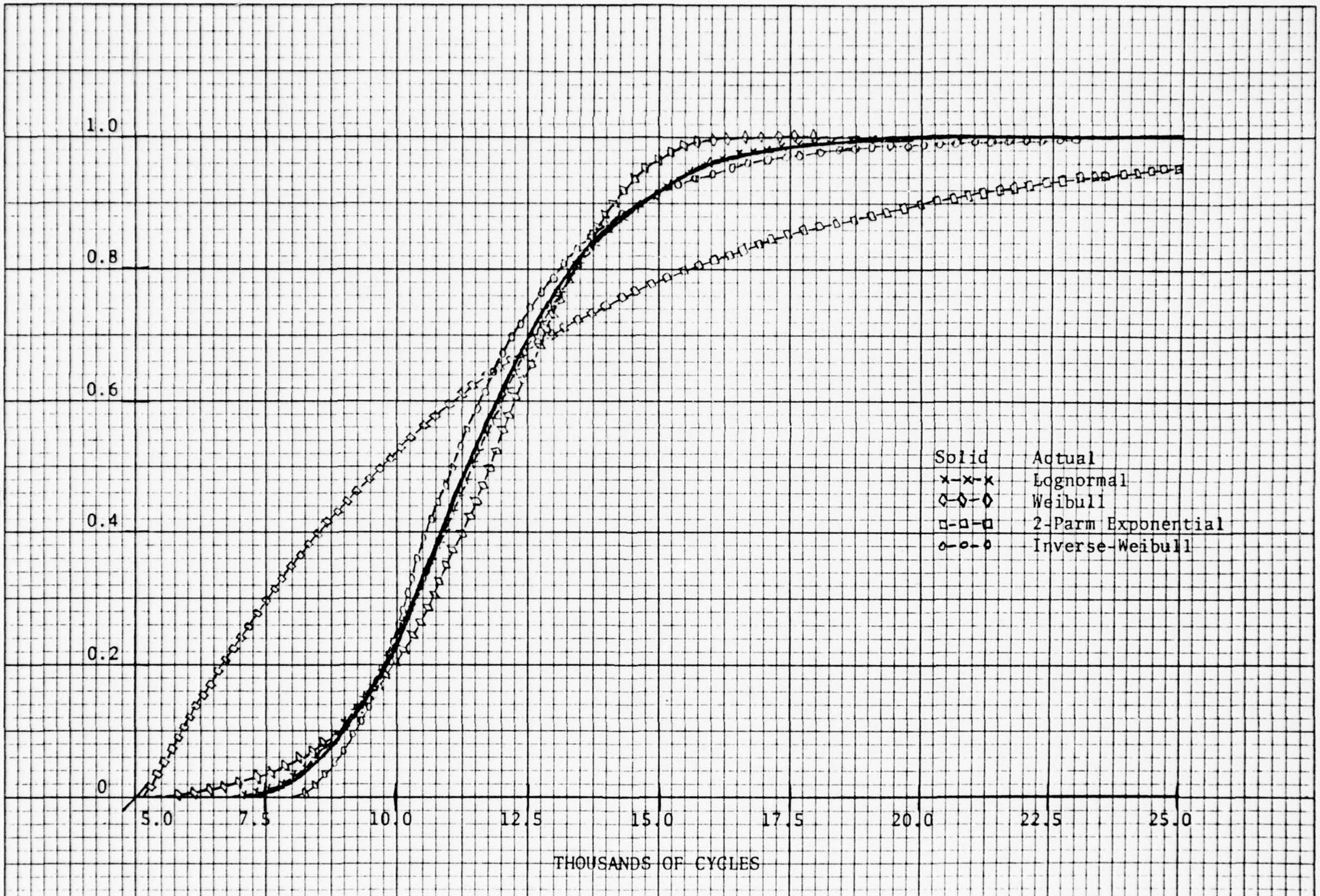

Figure 9. Cumulative distribution - Run NC1.

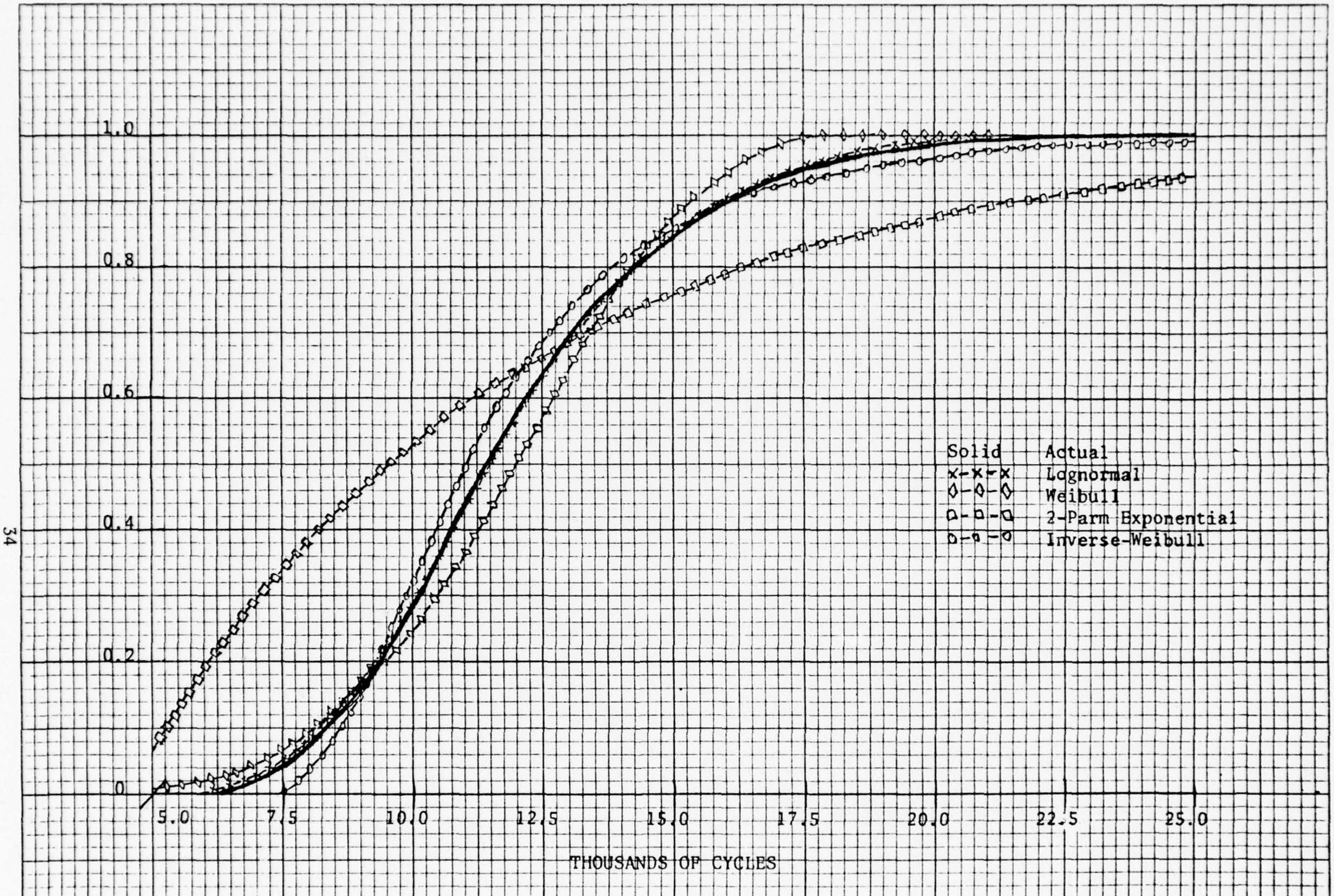

Figure 10. Cumulative distribution - Run NC2.

the cases when α is assumed to be uniform; Figure 12, when α is normal; and Figure 13, when α is extreme-value. Except for one of the dotted curves on Figure 13 (corresponding, incidentally, to run #3 discussed above), the dotted curves have approximately the same shape as the solid curves and tend to remain fairly close together.

8. SOME COMMENTS ON THE PROSCHAN-SETHURAMAN MODEL

Proschan and Sethuraman[2] have developed a model for gun tube fatigue also based on the Throop equation. They assume, as did Racicot, that none of the material parameters vary from cycle to cycle. They also assume, however, that none of the material properties vary from tube to tube. In the notation of this report, they assume that G is constant from tube to tube. They do assume that b_o varies *from tube to tube.*

Further, Proschan and Sethuraman assume that

a. A large number of initial cracks are present and the largest one present ultimately causes the failure.

b. The number of initial cracks has a Poisson distribution and the depths of the initial cracks are independent and identically distributed.

c. There exists some constant γ such that all initial crack lengths are no larger than γ, with probability 1.

---

[2]Proschan, F., and Sethuraman, J., "A Probabilistic Model for Initial Crack Size and Fatigue Life of Gun Barrels," Florida State University Technical Report No. M394, The Florida State University, Tallahassee, Florida, December 1976.

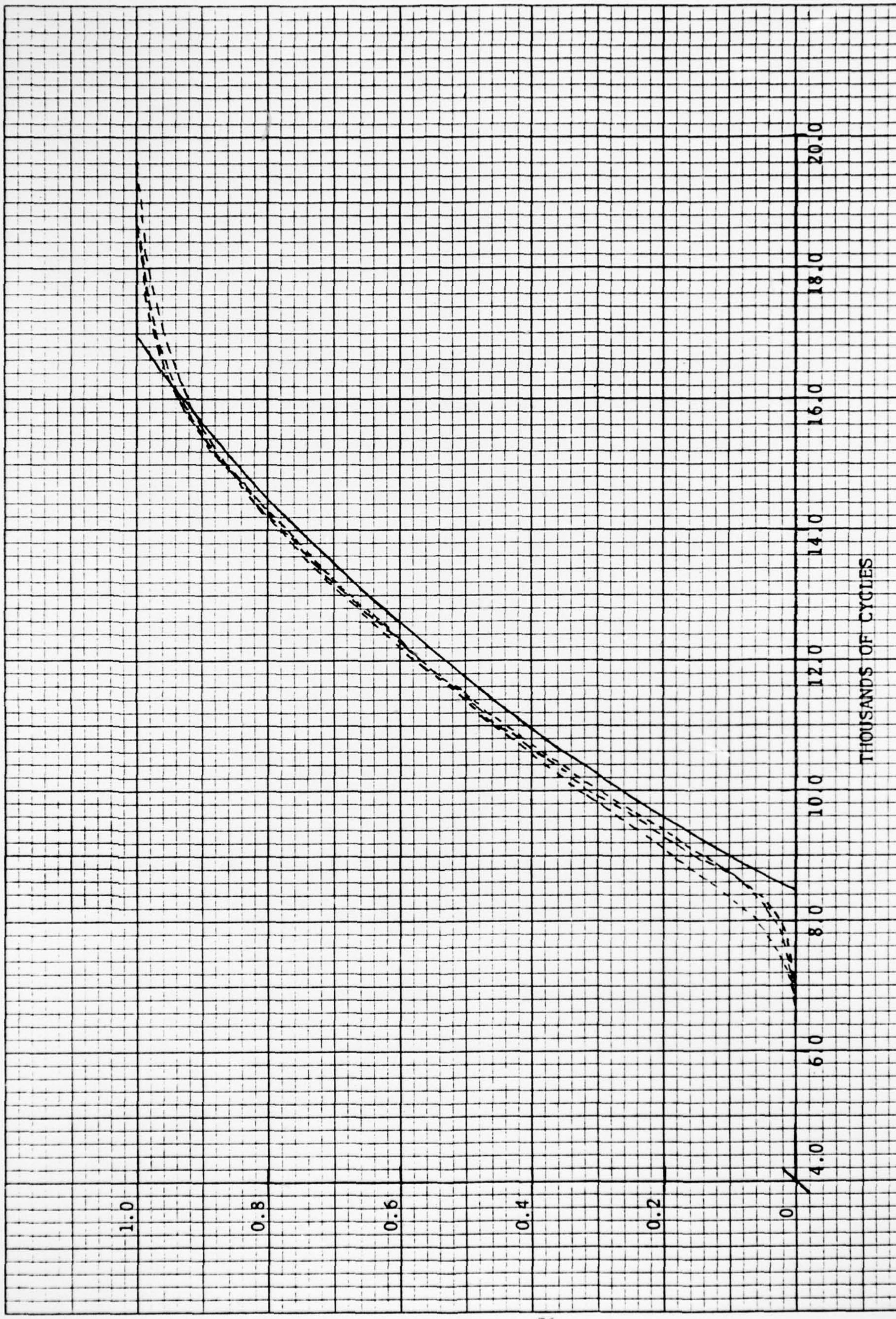

Figure 11. Cumulative distribution of fatigue life - effect of α (Uniform α).

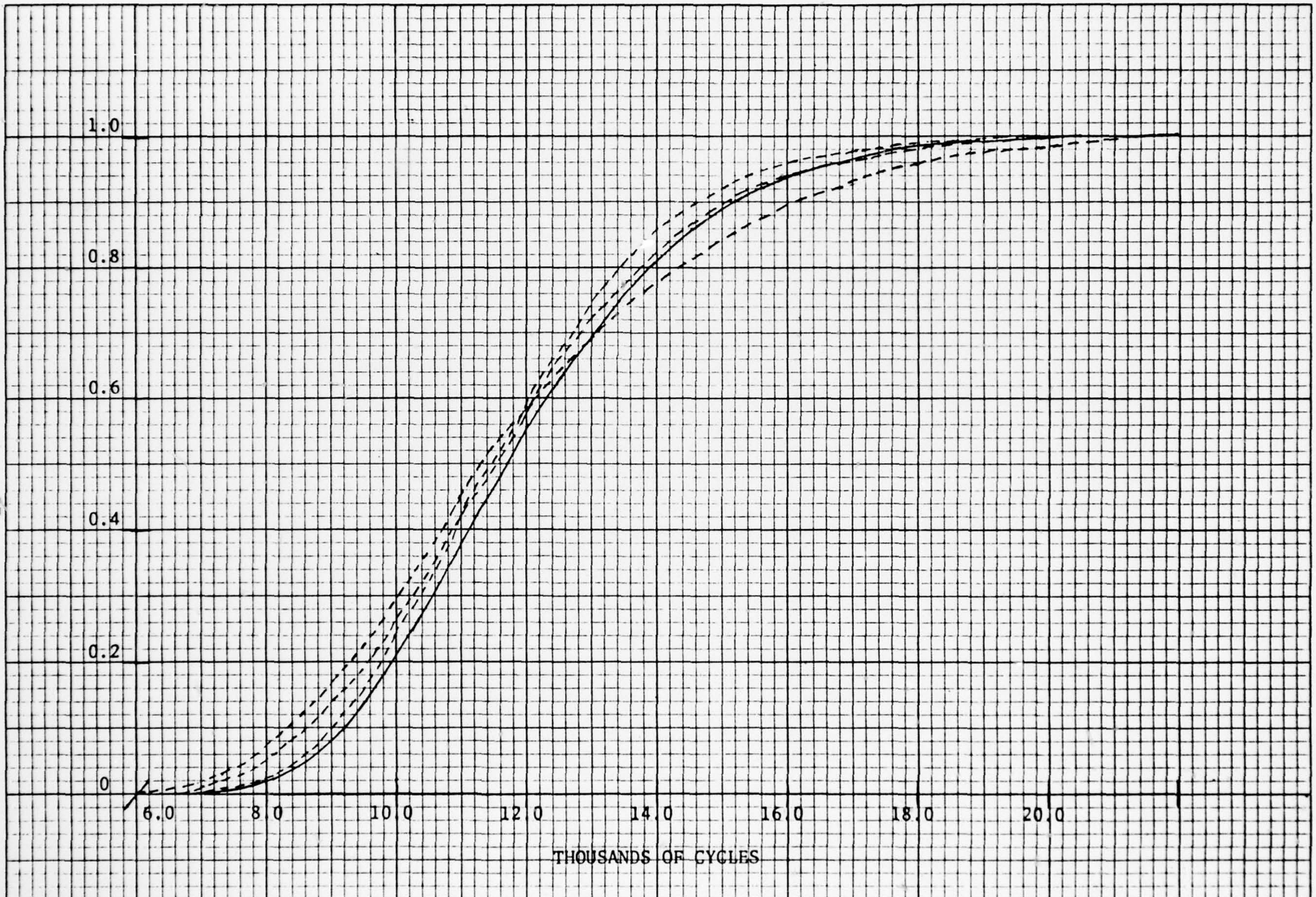

Figure 12. Cumulative distribution of fatigue life - effect of α (Normal α).

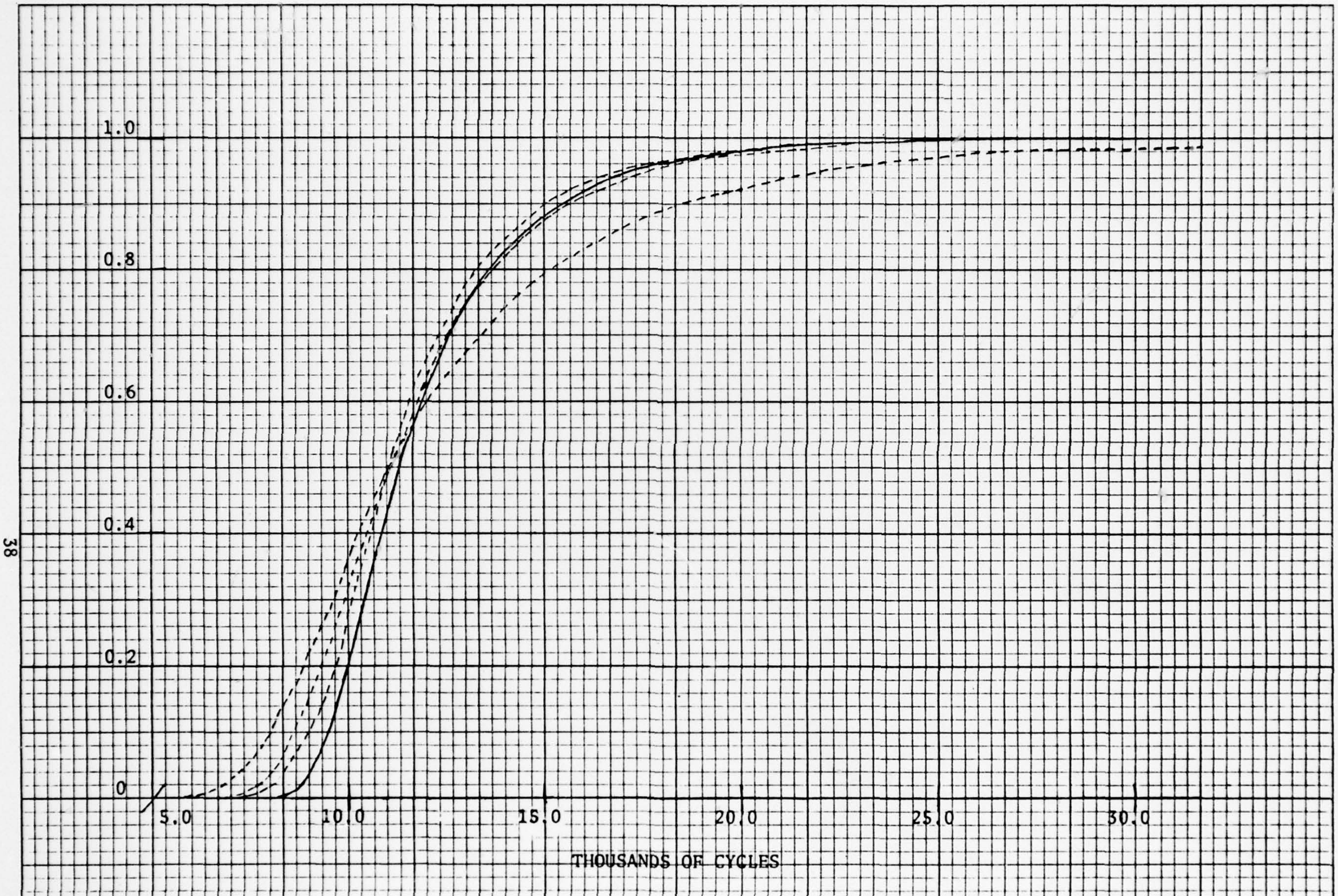

Figure 13. Cumulative distribution of fatigue life - effect of α (Extreme-Value α).

The present author finds assumptions a, b and c reasonable and cannot quibble with them. He can (and will) quibble with the assumption that all material properties are constant from tube to tube. From the results of the last section, it is clear that the variation in the material property α cannot be ignored.

Proschan and Sethuraman state without proof that "it can be shown from the theory of limiting distributions for the maximum (or minimum) of large numbers of random variables that the distribution of [fatigue life] can be approximated by a shifted Weibull distribution"[2]. It may be interesting to note that under their assumptions the fatigue life will approach a constant in probability (and hence also in distribution) as the average number of initial cracks becomes large. This is easily seen as follows: Let M be the number of initial cracks and let $D_1,\ldots,D_M$ be their depths. Let $D = \max(D_1,\cdots,D_M)$. By assumption a, fatigue life is a continuous function of D. So it suffices to show that D approaches a constant in probability. By assumption b, $D_1,\ldots,D_M$ have a common distribution. Denote its cdf by F. By assumption C, there exists some γ such that $F(x) = 1$ if $x > \gamma$ and $F(x) < 1$ if $x < \gamma$.

Now, for any x and $M \geq 1$,

$$Pr(D \leq x|M) = Pr(D_1 \leq x,\cdots,D_M \leq x|M) = F^M(x)$$

---

[2]Proschan, F., and Sethuraman, J., "A Probabilistic Model for Initial Crack Size and Fatigue Life of Gun Barrels," Florida State University Technical Report No. M394, The Florida State University, Tallahassee, Florida, December 1976, p. 6.

M has a Poisson distribution with mean λ. (λ is then the average number of initial cracks in a tube.) Then

$$\Pr(D \leq x) = \frac{\Pr(D \leq x | M \geq 1)}{\Pr(M \geq 1)}$$

$$= \frac{\sum_{M=1}^{\infty} e^{-\lambda}\lambda^M F^M(x)/M!}{\sum_{M=1}^{\infty} e^{-\lambda}\lambda^M/M!}$$

$$= \frac{e^{\lambda F(x)}-1}{e^{\lambda}-1}$$

$$= \frac{e^{-\lambda[1-F(x)]}-e^{-\lambda}}{1-e^{-\lambda}}$$

Now let $\varepsilon > 0$. Then $\Pr(D \geq \gamma + \varepsilon) = 0$. So

$$\Pr(|D-\gamma| \geq \varepsilon) = \Pr(D \leq \gamma-\varepsilon)$$

$$= \frac{e^{-\lambda[1-F(\gamma-\varepsilon)]}-e^{-\lambda}}{1-e^{-\lambda}} \tag{14}$$

But, $\gamma-\varepsilon < \gamma$ and so $1-F(\gamma-\varepsilon) > 0$. Then as $\lambda \to \infty$, it follows from equation (14) that $\Pr(|D-\gamma| \geq \varepsilon) \to 0$.

Hence, $D \to \gamma$ in probability as $\lambda \to \infty$.

9. CONCLUSIONS

Racicot concluded from his research that the lognormal gave the best fit to the fatigue life distribution. The present study shows that Racicot's conclusion holds up fairly well under conditions broader than he considered. The lognormal (and its close ally the Birnbaum-Saunders) almost always gave the best fit or ran a close second. However, it must be stressed that the lognormal can only be

an approximation. This is no reason to assume that the fatigue life distribution is lognormal. In fact, assumptions of lognormality could be rejected in most cases.

What is significant is that the lognormal performed much better than its principal rival, the Weibull, which has often been used in fatigue life studies. The two-parameter exponential, suggested by Proschan and Sethuraman, is not even close. In this regard, it is interesting to note that the distributions such as the inverse-Weibull, which to the author's knowledge has never been mentioned as a possibility for approximating the fatigue life distribution, did better than the much mentioned Weibull.

The author would have been quite delighted to be able to announce that a lognormal (or Weibull, or inverse-Weibull, etc.) was so close to the fatigue life distribution that it could be used with no further qualms. It does not appear at this time that any such announcement can be made. There would also seem to be little use quibbling as to whether the fatigue life distribution actually is lognormal or Weibull or exponential. It is most likely none of these.

At this point, a more useful approach would be to drop the attempt to find out what the fatigue life distribution really is, and instead to discover if the procedures derived from the candidate distributions give good results even though the actual fatigue life distribution is none of the candidates. In other words, we should examine such matters as to whether a confidenced safe life computed under an assumption of (say) lognormality for the fatigue lives gives a reasonable approximation

to a true confidenced safe life. Answering this type of question could ultimately be more fruitful than engaging over arguments over the true nature of the fatigue life distribution.

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APPENDIX

MATHEMATICAL DETAILS

This Appendix will deal with the mathematical details of the assertions made in Section 3, Case IV. For this we will need a slight generalization of the Central Limit Theorem.

<u>Theorem</u>: Let (x_n) be a sequence of independent, identically distributed random variables with $Ex_n = \mu$ and $Var(x_n) = \sigma^2$, where $0 < \sigma < \infty$. For each n, let $\alpha_{n1},\ldots,\alpha_{nn}$ be constants. Let (τ_n) and (μ_n) be sequences of constants such that:

a) $\frac{1}{\tau_n}\left(\sum_{j=1}^{n}\alpha_{nj}\mu-\mu_n\right) \to 0$ as $n \to \infty$

b) $\frac{1}{\tau_n^2}\sum_{j=1}^{n}\alpha_{nj}^2 \to 1$ as $n \to \infty$

c) $\max_{1\le j\le n}\left|\frac{\alpha_{nj}}{\tau_n}\right| \to 0$ as $n \to \infty$

Then $\frac{1}{\sigma\tau_n}\left(\sum_{j=1}^{n}\alpha_{nj}x_j - \mu_n\right)$ has an asymptotic standard normal distribution.

Proof: This theorem is proved essentially the same way as the standard Central Limit Theorem[4], which is the special case $\mu_n = \mu$. $\tau_n = 1/\sqrt{n}$, $\alpha_{nj} = 1/n$.

Let $\phi(u)$ be the common characteristic function of the random variables $(x_n - \mu)/\sigma$. Let $\phi_n(u)$ be the characteristic function of $\frac{1}{\sigma\tau_n}\left(\sum_{j=1}^{n}\alpha_{nj}x_j - \mu_n\right)$.

[4]Cramer, H., <u>Mathematical Methods of Statistics</u>, Princeton University Press, Princeton, NJ, 1946, p. 213. (The proof of the theorem is practically the same as Cramer's proof of the Central Limit Theorem, so many details have been omitted.)

Then

$$\phi(u) = 1 - \frac{u^2}{2} + \psi(u)$$

where $\psi(u)/u^2 \to 0$ as $u \to 0$, $u \neq 0$.

If $\xi(u) = \log\left[1 - \frac{u^2}{2} + \psi(u)\right] + \frac{u^2}{2}$, then $\xi(u)/u^2 \to 0$ as $u \to 0$.

Let $\beta_n = \frac{1}{\sigma\tau_n}\left(\sum_{j=1}^{n}\alpha_{nj}\mu - \mu_n\right)$. Then

$$\frac{1}{\sigma\tau_n}\left(\sum_{j=1}^{n}\alpha_{nj}x_j - \mu_n\right) = \beta_n + \sum_{j=1}^{n}\frac{\alpha_{nj}}{\tau_n}\left(\frac{x_n-\mu}{\sigma}\right).$$

So for any u,

$$\phi_n(u) = e^{iu\beta_n}\prod_{j=1}^{n}\phi\left(\frac{\alpha_{nj}u}{\tau_n}\right).$$

Then

$$\log\phi_n(u) = iu\beta_n + \sum_{j=1}^{n}\log\phi\left(\frac{\alpha_{nj}u}{\tau_n}\right)$$

$$= iu\beta_n - \frac{u^2}{2\tau_n^2}\sum_{j=1}^{n}\alpha^2{}_{nj} + \sum_{j=1}^{n}\xi\left(\frac{\alpha_{nj}u}{\tau_n}\right).$$

By assumption, $\beta_n \to 0$ and $\frac{1}{\tau_n^2}\sum_{j=1}^{n}\alpha^2{}_{nj} \to 1$. So it will suffice to show that $\sum_{j=1}^{n}\xi\left(\frac{\alpha_{nj}u}{\tau_n}\right) \to 0$ for any $u \neq 0$.

Fix u and let $\varepsilon > 0$. Then for some $\delta > 0$

$|\xi(x)| \leq \varepsilon\,|x|^2$ when $|x| < \delta$. For sufficiently large n, $|\alpha_{nj}/\tau_n| < \delta/|u|$ for all $1 \leq j \leq n$. Then for all j,

$\left|\xi\left(\frac{\alpha_{nj}u}{\tau_n}\right)\right| < \frac{\varepsilon u^2\alpha_{nj}{}^2}{\tau_n{}^2}$. Then

$$\left|\sum_{j=1}^{n}\xi\left(\frac{\alpha_{nj}u}{\tau_n}\right)\right| \leq \frac{\varepsilon u^2}{\tau_n{}^2}\sum_{j=1}^{n}\alpha^2{}_{nj} \tag{15}$$

But the right hand side of (15) will be close to εu^2 for large n.

Q.E.D.

Now assume the set-up as in Case IV, Section 3. There we had $S_N - NG_1 = \sum_{j=1}^{N-1} (N-j)\varepsilon_j$ where the ε_j's are independent and identically distributed with mean μ_ε and variance σ_ε^2. The theorem can now be applied with $\alpha_{nj} = n + 1 - j$ for $1 \leq j \leq n$, $\tau_n^2 = \frac{n(n+1)(2n+1)}{6}$ and $\mu_n = \frac{n(n+1)}{2}\mu$. The formulas[8]

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \text{ and } \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

may be used to verify that the conditions of the theorem hold. The conclusion is that

$$\frac{S_N - NG_1 - N(N-1)\mu\varepsilon/2}{[N(N-1)(2N-2)\sigma_\varepsilon^2/6]^{1/2}}$$

has an asymptotic standard normal distribution.

Now,

$$E(S_N) = N\mu_G + \sum_{j=1}^{N-1} (N-j)\mu_\varepsilon$$

$$= N\mu_G + \frac{N(N-1)}{2}\mu_\varepsilon \qquad (16)$$

If G_1 is independent of the ε's, then

$$\mathrm{Var}(S_N) = N^2\sigma_G^2 + \sum_{j=1}^{N-1} (N-j)^2\sigma_\varepsilon^2$$

$$= N^2\sigma_G^2 + \frac{N(N-1)(2N-1)}{6}\sigma_\varepsilon^2 \qquad (17)$$

Let $\tau_N^2 = \frac{N^2\sigma_G^2}{\sigma_\varepsilon^2} + \frac{N(N-1)(2N-1)}{6}$. Then $N^{3/2}/\tau_N \to \sqrt{3}$.

[8]Selby, S., Standard Mathematical Tables (14th Edition), The Chemical Rubber Co., Cleveland, OH, 1964, p. 390.

Then $\frac{N}{\sigma_\varepsilon \tau_N}$ $(G_1 - \mu_G) \to 0$ in probability, since $\frac{N}{\tau_N} = \frac{1}{N^{1/2}} \frac{N^{3/2}}{\tau_N} \to 0$.
Application of the theorem gives that $\frac{1}{\sigma_\varepsilon \tau_N} \left(\sum_{j=1}^{N-1} (N-j)\varepsilon_j - \frac{N(N-1)}{2} \mu_\varepsilon \right)$
has an asymptotic standard normal distribution. Then

$$\frac{S_N - N\mu_G - N(N-1)\mu_\varepsilon}{\sigma_\varepsilon \tau_N}$$

$$= \frac{N}{\sigma_\varepsilon \tau_N} (G_1 - \mu_G) + \frac{1}{\sigma_\varepsilon \tau_N} \left(\sum_{j=1}^{N-1} (N-j)\varepsilon_j - \frac{N(N-1)}{2} \mu_\varepsilon \right)$$

also has an asymptotic standard normal distribution[9]. This last result does not require independence of G_1 from the ε_j's, although equation (17) would not necessarily hold without such independence.

It is also possible to obtain asymptotic results not involving G_1 or its moments in any way. In fact if (τ_N) is any sequence such that $N^{3/2}/\tau_N \to \sqrt{3}$, then

a) $\frac{NG_1}{\sigma_\varepsilon \tau_N} \to 0$ in probability;

b) $\frac{1}{\sigma_\varepsilon \tau_n} \left(\sum_{j=1}^{N-1} (N-j)\varepsilon_j - \frac{N(N-1)}{2} \mu_\varepsilon \right)$ has an asymptotic standard normal distribution (by the Theorem).

Then

$$\frac{S_N - N(N-1)\mu_\varepsilon/2}{\sigma_\varepsilon \tau_N} = \frac{NG_1}{\sigma_\varepsilon \tau_N} + \frac{1}{\sigma_\varepsilon \tau_N} \left(\sum_{j=1}^{N-1} (N-j)\varepsilon_j - \frac{N(N-1)}{2} \mu_\varepsilon \right)$$

has an asymptotic standard normal distribution.

[9]Rao, C. R., Linear Statistical Inference and Its Applications, John Wiley & Sons, Inc. New York, NY, 1965, p. 102, Theorem (x).

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